Quantum Phase Transition in Ultracold ^{87}Rb Atom Gas with Radiant Field

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A second-order quantum phase transition in two-species Bose-Einstein condensates of ^{87}Rb atoms coupled by a quantized radiant field is revealed explicitly in terms of the energy spectrum which is obtained in the thermodynamic limit $1/N \to 0$ and is controllable by the coupling parameter between the atom and field. The scaling behavior of the collective excitation modes at the critical transition point is seen to be in the same universality class as that of the Dicke model. It is also demonstrated that the quantum phase transition is realizable below the critical temperature of BEC and can be detected experimentally by measuring the abrupt change of atom population imbalance.

Experimental realization of Bose-Einstein condensate (BEC) in trapped atomic gases has opened a possibility to explore the quantum mechanics of mesoscopic systems in a quantitatively new regime [1, 2, 3]. A paradigmatic effect in such systems is the coherent exchange of particles between weakly connected states, which was first predicted in superconductors [4] known as Josephson effect and is expected to be displayed also in trapped BECs by a double-well potential [5, 6, 7] or optical lattice [8, 9, 10]. In particular, recent experiments on two-component BECs of ^{87}Rb atoms have stimulated considerable interests in the phase dynamics and number fluctuations of the condensates [11, 12] in a radiant field which induces the coherent exchange of particles between two-species BECs. It has been demonstrated that the population oscillation between two components modulated by the collapses and revivals indicates the quantum nature of the system [13, 14]. The quantum system of two-level atoms coupled by a quantized field exhibits a novel phenomenon of super-radiance describing the collective emission of light from excited atoms and involving the spontaneous buildup of coherence in a macroscopic ensemble of atoms. Recently, great attentions have been attracted on super-radiant scattering of BECs consisting of atoms with two internal states both experimentally [15, 16]and theoretically [17, 18].

At zero-temperature all thermal fluctuations are frozen out whereas quantum fluctuations are dominative. The microscopic quantum fluctuations can give rise to a macroscopic phase transition in the ground state of many-body systems associated with the variation of a controllable parameter [19] which is now known as the quantum phase transition (QPT) originating from singularity of the energy spectrum [20]. Trapped BECs, which are well isolated from the environment, can be excited either by deforming the trap or by radiant field and thus are ideal quantum systems to observe the QPT. It has been shown that ultracold bosonic atoms loaded in an optical lattice possess a continuous quantum phase transition from a superfluid phase to a Mott insulator phase at low temperatures induced by varying the depth of the optical potential [21, 22]. While the super-radiant phase transition of BECs consisting of atoms with two internal states has not yet been investigated and may be of more fundamental importance.

In this letter we reveal a novel second-order quantum phase transition in the two-component BECs coupled by a radiant field with Raman coupling, which belongs to the same universality class as in the Dicke model describing the collective effects of N two-level atoms interacting with a single photon mode in quantum optics [23, 24, 25] where the atomic ensemble in the normal phase is collectively unexcited while is macroscopically excited with coherent radiations in a so-called super-radiant phase. As a matter of fact what we considered here is a generalized Dicke Hamiltonian with atom-atom interaction in addition. We obtain the ground state energy as a function of the coupling parameter between the atom and field which possesses a critical point at which the system undergoes a second-order phase transition. The QPT and the scaling behavior at the transition point are demonstrated explicitly by variation of the coupling parameter. Finally we evaluate the critical value of coupling parameter at phase transition point to show that the energy scale of it is realizable below the critical temperature of BEC and the super-radiant phase transition can be detected experimentally by measuring the atom population imbalance between two species.

We consider the ultracold ^{87}Rb atom gas consisting of two internal states $|F = 1, m_f = -1\rangle$ (denoting $|1\rangle$ for short) and $|F = 2, m_f = 1\rangle$ ($|2\rangle$) coupled by a radiant field in a single trap which applies a potential V_l ($l = 1, 2$) to the atoms of l-state. The interaction between atoms is considered as elastic two-body collision with the $\delta-$function-type potential. In the formalism of second quantization, Hamiltonian of this system with a quantized field reads

$$H = \sum_{l=1,2} H_l + H_{int} + H_{ed} + H_R \qquad (1)$$

with

$$H_l = \int d^3\mathbf{r}\{\Psi_l^+(\mathbf{r})[-\frac{\hbar^2}{2m}\Delta^2 + V_l(\mathbf{r})]\Psi_l(\mathbf{r})+$$
$$\frac{q_l}{2}\Psi_l^+(\mathbf{r})\Psi_l^+(\mathbf{r})\Psi_l(\mathbf{r})\Psi_l(\mathbf{r})\},$$

$$H_{int} = q_{1,2}\int d^3\mathbf{r}\Psi_1^+(\mathbf{r})\Psi_2^+(\mathbf{r})\Psi_1(\mathbf{r})\Psi_2(\mathbf{r}),$$

$$H_f = i\hbar\Omega\int d^3\mathbf{r}[\Psi_1^+(\mathbf{r})\Psi_2(\mathbf{r}) + \Psi_2^+(\mathbf{r})\Psi_1(\mathbf{r})](ae^{i\mathbf{k}\cdot\mathbf{r}} - e^{-i\mathbf{k}\cdot\mathbf{r}}a^+),$$

$$H_R = \omega a^+ a,$$

where $\Psi_l(\mathbf{r})$ is boson field operator of atoms in the internal atomic state-l. $q_l = 4\pi\hbar^2\rho_l/m$ measures the interaction between atoms of the same species while $q_{1,2} = 4\pi\hbar^2\rho_{1,2}/m$ denotes the atom-interaction of different species, where ρ_l and $\rho_{1,2} = \rho_{2,1}$ are the intraspecies $s-$wave scattering lengths and can be tuned by an external magnetic field via Feshbach resonance [26]. a, a^+ denote respectively the radiation-field annihilation and creation operators with frequency ω. H_f describes the electric dipole transition between two-species atoms induced by the radiation-field in the Schrödinger picture [27]. The interaction constant Ω is known as the Rabi frequency and $\mathbf{k}$ denotes the wave vector of the field. It should be noticed that the usual rotating wave approximation is not needed in our formulation of the interaction between atom and field.

In the single-mode approximation of condensate such that $\Psi_1(\mathbf{r}) = c_1\phi_1(\mathbf{r})$, $\Psi_2(\mathbf{r}) = c_2\phi_2(\mathbf{r})$, where c_1 and c_2 are the annihilation operators, the Hamiltonian (1) becomes

$$H = \sum_{l=1,2}(\omega_l c_l^+ c_l + \frac{\eta_l}{2}c_l^+ c_l^+ c_l c_l) + \chi c_1^+ c_1 c_2^+ c_2 + \tag{2}$$
$$\lambda_\Omega(c_1^+ c_2 + c_2^+ c_1)(a + a^+) + \omega a^+ a,$$

where $\omega_l = \int d^3\mathbf{r}\{\phi_l^*(\mathbf{r})[-\frac{\hbar^2}{2m}\Delta^2 + V_l(\mathbf{r})]\phi_l(\mathbf{r})$, $\eta_l = q_l\int d^3\mathbf{r}|\phi_l(\mathbf{r})|^4$, $\chi = q_{1,2}\int d^3\mathbf{r}|\phi_1(\mathbf{r})|^2|\phi_2(\mathbf{r})|^2$ and $\lambda_\Omega = -\hbar\Omega\int d^3\mathbf{r}\phi_1^*(\mathbf{r})\sin(\mathbf{k}\cdot\mathbf{r})\phi_2(\mathbf{r})$. Hamiltonian (2), which is the starting point of the paper, is a full quantum description of atom-condensate interacting with the radiant field. In terms of the pseudoangular momentum operators with the Schwinger relations defined as $S_x = \frac{1}{2}(c_1^+ c_2 + c_2^+ c_1)$, $S_y = \frac{1}{2i}(c_1^+ c_2 - c_2^+ c_1)$, and $S_z = \frac{1}{2}(c_1^+ c_1 - c_2^+ c_2)$, where the Casimir invariant is $S^2 = \frac{N}{2}(\frac{N}{2}+1)$ with N being the total atom number, Hamiltonian (2) is converted to $H = \omega a^+ a + \omega_0 S_z + q S_z^2 + 2\lambda_\Omega S_x(a^+ + a)$ with $\omega_0 = \omega_1 - \omega_2 + (N-1)(\eta_2 - \eta_1)/2$, $q = [(\eta_1 + \eta_2)/2 - \chi]$, which can be rewritten apart from a trivial constant as

$$H = \omega a^+ a + \tilde{\omega}_0 S_z + \frac{\lambda}{\sqrt{N}}(S_+ + S_-)(a^+ + a) + \frac{\nu}{N}S_+ S_-, \tag{3}$$

where $S_\pm = S_x \pm iS_y$, $\nu = -Nq$, $\lambda = \sqrt{N}\lambda_\Omega$ and $\tilde{\omega}_0 = \omega_0 + q \simeq \omega_0$ since the parameter q is a negligibly small number in our case seen below. The prefactor $1/N$ makes a finite free energy per atom in the thermodynamic limit $N \to \infty$.

We first of all derive the ground state energy spectrum as a function of the coupling parameter λ in order to reveal the QPT. To this end we use the Holstein-Primakoff transformation (HPT) of the angular momentum operators [28] $S_+ = b^+\sqrt{N - b^+ b}$, $S_- = \sqrt{N - b^+ b}\, b$ and $S_z = (b^+ b - N/2)$, where $[b, b^+] = 1$, to rewrite the Hamiltonian and subsequently introduce shifting boson operators $\tilde{a}^+$ and $\tilde{b}^+$ with properly scaled auxiliary parameters α and β such that $\tilde{a}^+ = a^+ + \sqrt{N}\alpha$ and $\tilde{b}^+ = b^+ - \sqrt{N}\beta$ [29]. We then expand the Hamiltonian expressed with the new boson operators $\tilde{a}^+$ and $\tilde{b}^+$ as power series of $1/\sqrt{N}$ and obtain $H = NH_0 + N^{1/2}H_1 + N^0 H_2 + \cdots$, the first term of which gives rise to the Hartree-Bogoliubov ground state energy [30]

$$E_0 = \begin{cases} -N\omega_0/2, & \lambda < \lambda_c \\ -N[(\frac{\lambda^2}{\omega} - \frac{\nu}{4})(1 - \delta^2) + \frac{\omega_0\delta}{2}], & \lambda > \lambda_c \end{cases}, \tag{4}$$

where $\delta = \omega\omega_0/(4\lambda^2 - \omega\nu)$ and

$$\lambda_c = \frac{1}{2}\sqrt{\omega(\omega_0 + \nu)} \text{ or } \lambda_{\Omega,c} = \frac{1}{2}\sqrt{\omega(\omega_0 + \nu)/N}. \tag{5}$$

The auxiliary parameters α and β are determined from minimizing the ground state energy and thus are given by

$$\alpha = \begin{cases} 0 & , \lambda < \lambda_c \\ \frac{\lambda}{\omega}\sqrt{1 - \delta^2} & , \lambda > \lambda_c \end{cases},$$

$$\beta = \begin{cases} 0 & , \lambda < \lambda_c \\ \sqrt{(1 - \delta)/2} & , \lambda < \lambda_c \end{cases}. \tag{6}$$

The ground state energy as a function of the coupling parameter is shown in Fig. 1 which indicates the typical second-order phase transition at the transition point λ_c. The QPT is similar to that observed in the Dicke Hamiltonian [25] and the two phases may be respectively called the normal ($\lambda < \lambda_c$) and super-radiant ($\lambda > \lambda_c$) phases according to the properties of the ground state and the atom population imbalance shown below.

Having determined the transition point λ_c we now study the critical behavior of the system at the phase transition point. For $\lambda < \lambda_c$, $\alpha = \beta = 0$, we may take the approximation $\sqrt{N - b^+b} \simeq \sqrt{N}$ in the HPT of the angular momentum operators and the effective Hamiltonian is seen to be $H_< = \omega a^+a + (\omega_0 + \nu)b^+b + \lambda(b + b^+)(a^+ + a) - \frac{N\omega_0}{2}$, which is bilinear in the bosonic operators and can thus be diagonalized in general with the Bogoliubov transformation. Here we adopt an alternative approach in terms of the canonical operators $a^+ = (\sqrt{\omega}x - ip_x/\sqrt{\omega})/\sqrt{2}$, $b^+ = (\sqrt{\omega_0 + \nu}y - ip_y/\sqrt{\omega_0 + \nu})/\sqrt{2}$ and the Hamiltonian, in a rotating coordinate system such that $x = q_1 \cos\gamma + q_2 \sin\gamma$, $y = -q_1 \sin\gamma + q_2 \cos\gamma$ with $\gamma = \frac{1}{2}\arctan 4\lambda\sqrt{\omega(\omega_0 + \nu)}/[(\omega_0 + \nu)^2 - \omega^2]$, has the simple form

$$H_{(<)} = \frac{1}{2}[E_{(<)-}^2 q_1^2 + p_1^2 + E_{(<)+}^2 q_2^2 + p_2^2 - \omega_0 - \omega - \nu] - \frac{N\omega_0}{2}, \tag{7}$$

where the collective modes of fundamental excitations

$$E_{(<)\pm}^2 = \frac{1}{2}\{[\omega^2 + (\omega_0 + \nu)^2] \pm \sqrt{[\omega^2 - (\omega_0 + \nu)^2]^2 + 16\lambda^2\omega(\omega_0 + \nu)}\} \tag{8}$$

are recognized as the atomic and photonic modes respectively. The λ−dependence property of photonic mode $E_{(<)-}$ is shown in Fig.2. When the effective coupling strength λ approaches the critical value λ_c, the photonic mode $E_{(<)-}$ vanishes as

$$E_{(<)-}(\lambda \to \lambda_c) \sim \sqrt{\frac{32\lambda_c^3\omega^2}{16\lambda_c^4 + \omega^4}}|\lambda - \lambda_c|^{\frac{1}{2}}. \tag{9}$$

Above the phase transition point ($\lambda > \lambda_c$) a similar effective Hamiltonian in terms of canonical operators P and Q is found as

$$H_{(>)} = \frac{1}{2}[E_{(>)-}^2 Q_1^2 + P_1^2 + E_{(>)+}^2 Q_2^2 + P_2^2 - \omega_0 - \omega_1] \tag{10}$$
$$- N[(\frac{\lambda^2}{\omega} - \frac{\nu}{4})(1 - \delta^2) + \frac{\omega_0\delta}{2}]$$

with the corresponding atomic and photonic modes

$$E_{(>)\pm}^2 = \frac{1}{2}\{[\omega^2 + \frac{\omega_1^2 (\omega_1 + 2\omega_3 + 4\omega_4)}{\omega_1 - 2\omega_3}]$$
$$\pm \sqrt{[\omega^2 - \frac{\omega_1^2 (\omega_1 + 2\omega_3 + 4\omega_4)}{\omega_1 - 2\omega_3}]^2 + \frac{16\omega_1^2\omega_2^2\omega}{\omega_1 - 2\omega_3}}\} \tag{11}$$

(see Fig.2) where $\omega_1 = \frac{4\lambda^2}{\omega} - \nu(1 - \delta)$, $\omega_2 = \lambda\delta\sqrt{\frac{2}{(1+\delta)}}$, $\omega_3 = \frac{\lambda^2}{\omega}(1 - \delta) - \frac{\nu}{2}(1 - \delta)$, $\omega_4 = \frac{\lambda^2}{\omega}\frac{(1-\delta)^2}{2(1+\delta)}$. The scaling behavior of the photonic mode $E_{(>)-}$ at the critical point λ_c is the same as that in the normal phase as it should be.

The scaling behavior of the photonic mode, $E_- \propto |\lambda - \lambda_c|^{\frac{1}{2}}$, with the exponent describing a characteristic divergence length $\xi = (E_-)^{-1/2}$ [19]. The energy of the atomic mode E_+ tends to a value of $\sqrt{\omega^2 + (\omega_0 + \nu)^2}$ as $\lambda \to \lambda_c$ from both sides.

The atom population imbalance between two internal states of ^{87}Rb atoms in the ground state, $\Delta N = N_1 - N_2 = 2\langle S_z \rangle$, is found as

$$\frac{\Delta N}{N} = \begin{cases} -1, & \lambda < \lambda_c \\ -\omega\omega_0/(4\lambda^2 - \omega\nu), & \lambda > \lambda_c \end{cases} , \tag{12}$$

which as a function of the coupling strength λ is plotted in Fig.(3). In the normal phase ($\lambda < \lambda_c$) atomic ensemble is essentially in the lower energy state, whereas acquires macroscopic excitation above the transition point. This QPT may be observed by measuring the abrupt change of the atom population imbalance over increasing the Rabi frequency in practical experiments. In practice the abrupt change of the atom population imbalance can be measured experimentally by the variation of absorption spectrum of the light with respect to the coupling parameter λ_{Ω_c}. Then a critical point is to see whether or not the critical value of coupling parameter $\lambda_{\Omega_c} = \frac{\lambda_c}{\sqrt{N}}$ is realizable in BEC experiments of ^{87}Rb atoms. For the magnetic trap of an axial frequency $\omega_z = 69$ Hz and radial frequency $\omega_{x,y} = 21$ Hz, the $s-$wave scattering lengths are $\rho_1 = 5.36$ nm, $\rho_2 = 5.70$ nm, $\rho_{1,2} = \rho_{2,1} = 5.53$ nm, and the number of trapped atoms is $N = 5 \times 10^5$ [12]. The parameter ω_0 is found approximately as $\omega_0 = 4.1 \times 10^2$ Hz. The laser frequency used by Ketterle'group for Rayleigh scattering is $\omega = 1.7$ GHz [15]. With these experimental parameters the energy scale of the critical coupling parameter at transition point is evaluated as $\lambda_{\Omega,c} = 4.47$ nk, which is far below the critical temperature $T_c = 150$ nk of dilute ^{87}Rb gas BEC and thus the quantum phase transition is experimentally realizable.

In conclusion, we have demonstrated a second-order quantum phase transition in two-species BECs coupled by a radiant field with Raman coupling similar to the well known transition in the Dicke model of quantum optics where there is no atom-atom interaction. The quantum phase transition is more natural in BEC and can be observed directly by measuring the atom population imbalance between the two components of the BECs.

[1] ANDERSON H. M. et al., Science **269** (1995), 198.
[2] DAVIS K. *et al.*, Phys. Rev. Lett. **75** (1995), 3969.
[3] BRADLEY C. C., SACKETT C. A. and HULET R. G., Phys. Rev. Lett. **78** (1997) 985.
[4] JOSEPHSON B. D., Phys. Lett. **1** (1962), 251.
[5] SMERZI A., Phys. Rev. Lett. **79** (1997) 4950.
[6] ZAPATA I., SOLS F. and LEGGET A. J. Phys. Rev. A **57** (1998), R28.
[7] MARINO I. *et al.*, Phys. Rev. A **60** (1999), 487.
[8] ANDERSON B. P. and KASEVICH M. A. Science **282** (1998), 1686.
[9] BLOCH I. *et al.*, Nature (London) **403** (2000), 166.
[10] CATALIOTTI F. S. *et al.*, Science **293** (2001), 843.
[11] ANDREWS M. R. *et al.*, Science **275** (1997), 637.
[12] HALL D.S. *et al.*, Phys. Rev. Lett. **81** (1998), 1543.
[13] LEGGET A. J., Rev. Mod. Phys. **73** (2001), 307.
[14] CHEN Z. D. *et al.*, Phys. Rev. A **69 (2004)**, 023611.
[15] INOUYE *et al.*, Science **285** (1999), 571.
[16] SCHNEBLE *et al.,* Science **300** (2003), 475.
[17] kETTERLE W. and INOUYE S., Phys. Rev. Lett. **86** (2001), 4203.
[18] MOORE M. G. and MEYSTRE P., Phys. Rev. Lett. **86** (2001), 4199.
[19] SACHDEV S., *Quantum Phase transtitions* (Cambridge University Press, Cambridge, 1999).
[20] YANG C. N. and LEE T. D., Phys. Rev. **87** (1952), 410.
[21] JAKSCH D. *et al.*, Phys. Rev. Lett. **81** (1998), 3108.
[22] GREINER M. *et al.*, Nature (London) **415** (2002), 39.
[23] DICKE R. H., Phys. Rev. **93** (1954), 99.
[24] GROSS M. and MAROCHE S., Phys. Rep. **93** (1982), 301.
[25] EMARY C. and BRANDES T., Phys. Rev. E **67** (2003), 066203.
[26] FESHBACH H., *Theoretical Nuclear Physics* (Wiley, New York, 1992).
[27] lOUDON R, *The Quantum Theory of Light* (Oxford University Press, 2nd ed, 1983).

[28] HOLSTEIN T. and PRIMAKOFF H., Phys. Rev. **58** (1949), 1098.

[29] The displacements, $\widetilde{a}^+ = a^+ - \sqrt{N}\alpha$ and $\widetilde{b}^+ = b^+ + \sqrt{N}\beta$, also lead to the same result except the change of sign.

[30] DZHIOEV A. *et. al.*, Phys. Rev. C **69** (2004), 014318.

Figure Captions

Fig.1 The scaled ground state energy versus the coupling strength λ (in unit $10^6 \hbar/s$) with $\nu = 0$.

Fig.2 The photonic mode $E_{(<)-}$ (in unit $10^4 \hbar/s$) as a function of coupling strength λ (in unit $10^6 \hbar/s$) with $\omega_0 = 4.1 \times 10^2$ Hz, $\omega = 1.7 \times 10^9$ Hz.

Fig.3 The atom population imbalance between two internal states of ^{87}Rb atoms versus the coupling strength λ.